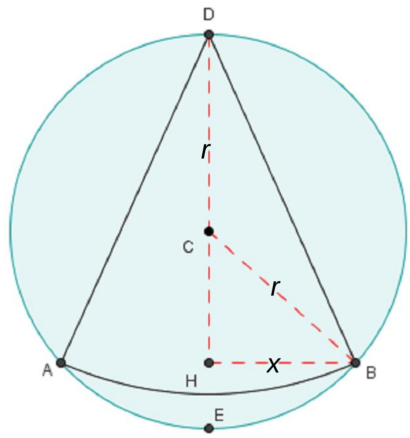
In una sfera di raggio $r$ inscrivere il cono avente il volume massimo.




*Soluzione 1*

Posto $\overline{BH} = x$ con $x \in [0,\ r]$

si ha: $\overline{CH} = \sqrt{r^2 - x^2}$ e $\overline{DH} = r + \sqrt{r^2 - x^2}$ .

La funzione da rendere massima è:

$V(x) = \frac{1}{3} \cdot S_{base} \cdot h$ ; $V(x) = \frac{1}{3}\pi x^2 \cdot \left( r + \sqrt{r^2 - x^2} \right)$

Agli estremi $x = 0$ e $x = r$ il cono degenera in un segmento *(il diametro)* avente volume nullo.

La derivata prima è: $V'(x) = \frac{\pi}{3} \cdot \left[ 2x \cdot \left( r + \sqrt{r^2 - x^2} \right) + x^2 \cdot \frac{-2x}{2\sqrt{r^2 - x^2}} \right] =$

$= \frac{\pi}{3} \cdot x \cdot \left[ 2 \cdot \left( r + \sqrt{r^2 - x^2} \right) + \frac{-x^2}{\sqrt{r^2 - x^2}} \right] = \frac{\pi}{3} x \cdot \frac{2\sqrt{r^2 - x^2} \cdot \left( r + \sqrt{r^2 - x^2} \right) - x^2}{\sqrt{r^2 - x^2}} =$

$= \frac{\pi}{3} x \cdot \frac{2r\sqrt{r^2 - x^2} + 2 \cdot \left( r^2 - x^2 \right) - x^2}{\sqrt{r^2 - x^2}} = \frac{\pi}{3} x \cdot \frac{2r\sqrt{r^2 - x^2} + 2r^2 - 3x^2}{\sqrt{r^2 - x^2}}$ .

$V'(x) = 0$ ; $\frac{\pi}{3} x \cdot \frac{2r\sqrt{r^2 - x^2} + 2r^2 - 3x^2}{\sqrt{r^2 - x^2}} = 0$ ; $\begin{cases} x = 0 \\ 2r\sqrt{r^2 - x^2} + 2r^2 - 3x^2 = 0 \quad (*) \end{cases}$

(*) $2r\sqrt{r^2 - x^2} + 2r^2 - 3x^2 = 0$ ; $2r\sqrt{r^2 - x^2} = 3x^2 - 2r^2$ ; $\left( 2r\sqrt{r^2 - x^2} \right)^2 = \left( 3x^2 - 2r^2 \right)^2$ ;

$4r^2 \cdot \left( r^2 - x^2 \right) = 9x^4 + 4r^4 - 12r^2x^2$ ; $4r^4 - 4r^2x^2 - 9x^4 - 4r^4 + 12r^2x^2 = 0$ ; $-9x^4 + 8r^2x^2 = 0$ ;

$x^2 \cdot \left( -9x^2 + 8r^2 \right) = 0$ ; $\begin{cases} x^2 = 0 \\ -9x^2 + 8r^2 = 0 \end{cases}$ $\begin{array}{l} x = 0 \ \text{doppia} \\ x = \mp \frac{2\sqrt{2}}{3} r \ \ \text{la soluzione negativa non è accettabile} \end{array}$

Essendo:

$V(0) = \frac{1}{3}\pi\, 0^2 \cdot \left( r + \sqrt{r^2 - 0^2} \right) = 0$

$V(r) = \frac{1}{3}\pi\, r^2 \cdot \left( r + \sqrt{r^2 - r^2} \right) = 0$

$V\left( \frac{2\sqrt{2}}{3} r \right) = \frac{1}{3}\pi \left( \frac{2\sqrt{2}}{3} r \right)^2 \cdot \left( r + \sqrt{r^2 - \left( \frac{2\sqrt{2}}{3} r \right)^2} \right) =$

$= \frac{1}{3}\pi \frac{8}{9} r^2 \cdot \left( r + \sqrt{r^2 - \frac{8}{9} r^2} \right) = \frac{8}{27}\pi\, r^2 \cdot \left( r + \frac{1}{3} r \right) =$

$= \frac{8}{27}\pi\, r^2 \cdot \frac{4}{3} r = \frac{32}{81}\pi\, r^3$

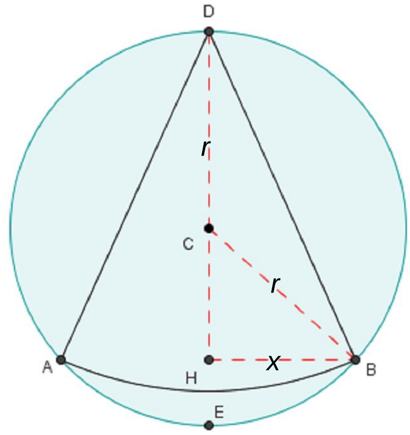
$\Rightarrow$

Il massimo assoluto è $M = \frac{32}{81}\pi\, r^3$

assunto nel punto $x = \frac{2\sqrt{2}}{3} r$

Si può concludere quindi che il cono avente il volume massimo ha il raggio di base uguale a $x = \frac{2\sqrt{2}}{3} r$ .

## *Soluzione 2*




Posto $\overline{BH} = x$ con $x \in [0,\ r]$

si ha: $\overline{CH} = \sqrt{r^2 - x^2}$ e $\overline{DH} = r + \sqrt{r^2 - x^2}$ .

La funzione da rendere massima è:

$$V(x) = \frac{1}{3} \cdot S_{base} \cdot h = \frac{1}{3} \pi x^2 \cdot \left( r + \sqrt{r^2 - x^2} \right)$$

Agli estremi $x = 0$ e $x = r$ il cono degenera in un segmento *(il diametro)* avente volume nullo.

La derivata prima è: $V'(x) = \frac{\pi}{3} \cdot \left[ 2x \cdot \left( r + \sqrt{r^2 - x^2} \right) + x^2 \cdot \frac{-2x}{2\sqrt{r^2 - x^2}} \right] =$

$$= \frac{\pi}{3} \cdot x \cdot \left[ 2 \cdot \left( r + \sqrt{r^2 - x^2} \right) + \frac{-x^2}{\sqrt{r^2 - x^2}} \right] = \frac{\pi}{3} x \cdot \frac{2\sqrt{r^2 - x^2} \cdot \left( r + \sqrt{r^2 - x^2} \right) - x^2}{\sqrt{r^2 - x^2}} =$$

$$= \frac{\pi}{3} x \cdot \frac{2r\sqrt{r^2 - x^2} + 2 \cdot \left( r^2 - x^2 \right) - x^2}{\sqrt{r^2 - x^2}} = \frac{\pi}{3} x \cdot \frac{2r\sqrt{r^2 - x^2} + 2r^2 - 3x^2}{\sqrt{r^2 - x^2}} .$$

$$V'(x) = 0 ; \quad \frac{\pi}{3} x \cdot \frac{2r\sqrt{r^2 - x^2} + 2r^2 - 3x^2}{\sqrt{r^2 - x^2}} = 0 ; \quad \begin{cases} x = 0 \\ 2r\sqrt{r^2 - x^2} + 2r^2 - 3x^2 = 0 \quad (*) \end{cases}$$

$$(*) \quad 2r\sqrt{r^2 - x^2} + 2r^2 - 3x^2 = 0 ; \quad 2r\sqrt{r^2 - x^2} = 3x^2 - 2r^2 ; \quad \left( 2r\sqrt{r^2 - x^2} \right)^2 = \left( 3x^2 - 2r^2 \right)^2 ;$$

$$4r^2 \cdot \left( r^2 - x^2 \right) = 9x^4 + 4r^4 - 12r^2x^2 ; \quad 4r^4 - 4r^2x^2 - 9x^4 - 4r^4 + 12r^2x^2 = 0 ; \quad -9x^4 + 8r^2x^2 = 0 ;$$

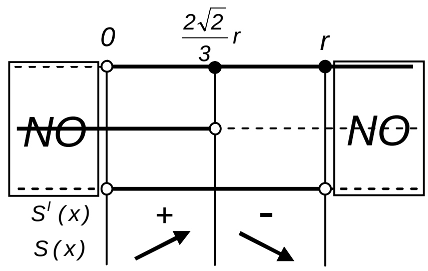
$$x^2 \cdot \left( -9x^2 + 8r^2 \right) = 0 ; \quad \begin{cases} x^2 = 0 \\ -9x^2 + 8r^2 = 0 \end{cases} \quad \begin{matrix} x = 0 \ \ doppia \\ x = \mp \frac{2\sqrt{2}}{3} r \end{matrix}$$

$$V'(x) > 0 ; \quad \frac{\pi}{3} x \cdot \frac{2r\sqrt{r^2 - x^2} + 2r^2 - 3x^2}{\sqrt{r^2 - x^2}} > 0 ;$$

$x > 0$ $\qquad$ $x > 0$

$2r\sqrt{r^2 - x^2} + 2r^2 - 3x^2 > 0 \quad (**)$ $\qquad$ $x < \frac{2\sqrt{2}}{3} r$

$\sqrt{r^2 - x^2} > 0$ $\qquad$ $-r < x < +r$



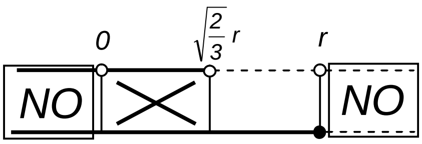

Si può concludere quindi che il cono avente il volume massimo ha il raggio di base uguale a $x = \frac{2\sqrt{2}}{3} r$ .

Calcoli

La disequazione *(**)* $2r\sqrt{r^2-x^2}+2r^2-3x^2>0\,;\quad 2r\sqrt{r^2-x^2}>3x^2-2r^2\,;\quad \sqrt{r^2-x^2}>\frac{3}{2r}x^2-r\,;$

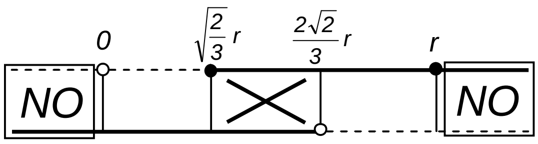
La soluzione di tale disequazione è data dall'unione dei sistemi: A $\begin{cases}\frac{3}{2r}x^2-r<0\\ r^2-x^2\ge 0\end{cases}$ $\bigcup$ B $\begin{cases}\frac{3}{2r}x^2-r\ge 0\\ r^2-x^2>\left(\frac{3}{2r}x^2-r\right)^2\end{cases}$

A $\begin{cases}\frac{3}{2r}x^2-r<0\\ r^2-x^2\ge 0\end{cases}$ $\begin{cases}-\sqrt{\frac{2}{3}}\,r<x<+\sqrt{\frac{2}{3}}\,r\\ -r\le x\le +r\end{cases}$




$0<x<+\sqrt{\frac{2}{3}}\,r$

B $\begin{cases}\frac{3}{2r}x^2-r\ge 0\\ r^2-x^2>\left(\frac{3}{2r}x^2-r\right)^2\end{cases}$ $\begin{cases}x\le-\sqrt{\frac{2}{3}}\,r\,;\quad x\ge+\sqrt{\frac{2}{3}}\,r\\ 0<x<\frac{2\sqrt{2}}{3}r \qquad (\#)\end{cases}$



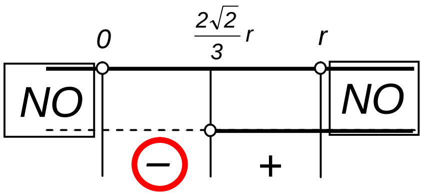

$\sqrt{\frac{2}{3}}\,r\le x<\frac{2\sqrt{2}}{3}r$

Pertanto *(**)* $2r\sqrt{r^2-x^2}+2r^2-3x^2>0$ per $x<\frac{2\sqrt{2}}{3}r$ .

Nota *(#)* $r^2-x^2>\left(\frac{3}{2r}x^2-r\right)^2\,;\quad r^2-x^2>\frac{9}{4r^2}x^4+r^2-3x^2\,;\quad \frac{9}{4r^2}x^4-2x^2<0\,;\quad x^2\cdot\left(\frac{9}{4r^2}x^2-2\right)<0\,;$

$x^2>0 \qquad \forall x\ne 0$

$\frac{9}{4r^2}x^2-2>0 \qquad x<-\frac{2\sqrt{2}}{3}r\,;\quad x>+\frac{2\sqrt{2}}{3}r$




$0<x<\frac{2\sqrt{2}}{3}r$