XVII NATIONAL MATHEMATICAL CONTEST

Cesenatico, May 4th, 2001

1. In a hexagon with all equal angles, the lengths of four consecutive edges are 5, 3, 6 and 7 (in this order). Find the lengths of the remaining two edges.

2. In a basket tournament every team plays twice versus each other team. A single team wins the tournament with 26 points and exactly two teams are in the last position with 20 points. How many teams participated in the tournament?

   (Recall that a won game gives 2 points, a lost one gives 0 points and no game can finish with a draw.)

3. Given the equation $x^{2001} = y^x$,

   (a) find all pairs (x, y) of solutions with x prime and y a positive integer;

   (b) find all pairs (x, y) of positive integers satisfying the equation.

   (Recall that $2001 = 3 \cdot 23 \cdot 29$)

4. Call *monotone* a positive integer such that in its decimal representation

   – there are at least two digits;
   – all digits are different from zero;
   – the digits appear in a strictly increasing or strictly decreasing order.

   (E. g. 127 and 9742 are monotone, whereas 172, 1224 and 7320 are not.)

   (a) Compute the sum of all monotone numbers having five digits.

   (b) Determine the number of final zeroes of the decimal representation of the least common multiple of all monotone numbers (without any restiction on the number of digits).

5. The incircle γ of triangle ABC touches the side AB at T. Let D be the point on γ diametrically opposite to T and let S be the intersection of the line through C and D with the side AB.

   Show that $AT = SB$.

6. A square is filled with 100 lamps, ranged in 10 rows and 10 columns. Some of them are alight, the others are out.

   To each lamp corresponds a push-button that, when pressed, switches all lamps of its row and its column (including the lamp itself).

   (a) Determine the states from which it is possible to light all the lamps.

   (b) Which is the answer if the the square has 81 lamps in 9 rows and 9 columns?