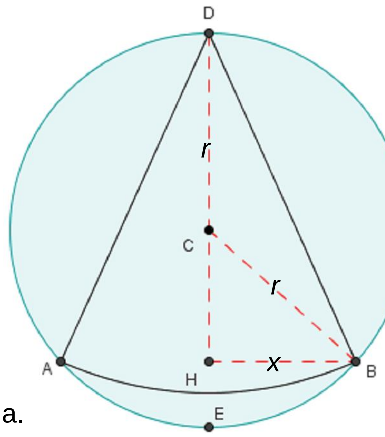
In una sfera di raggio r inscrivere il cono avente la superficie laterale massima.



<u>*Soluzione 1*</u>

Posto $\overline{BH} = x$ con $x \in [0,\ r]$ si ha: $\overline{CH} = \sqrt{r^2 - x^2}$ $\quad \overline{DH} = r + \sqrt{r^2 - x^2}$

$$\overline{BD} = \sqrt{\overline{DH}^2 + \overline{BH}^2} = \sqrt{\left(r + \sqrt{r^2 - x^2}\right)^2 + x^2}$$

$$= \sqrt{r^2 + r^2 - x^2 + 2r\sqrt{r^2 - x^2} + x^2} = \sqrt{2r^2 + 2r\sqrt{r^2 - x^2}}\ .$$

La funzione da rendere massima è:

$$S(x) = \frac{1}{2} \cdot 2\pi \cdot \overline{BH} \cdot \overline{BD}\,; \qquad \boxed{S(x) = \pi\, x \cdot \sqrt{2r^2 + 2r\sqrt{r^2 - x^2}}}$$

Agli estremi $x = 0$ e $x = r$ il cono degenera nel diametro avente superficie nulla.

La derivata prima è: $$S'(x) = \pi \cdot \left[1 \cdot \sqrt{2r^2 + 2r\sqrt{r^2 - x^2}} + x \cdot \frac{2r \cdot \dfrac{-2x}{2\sqrt{r^2 - x^2}}}{2\sqrt{2r^2 + 2r\sqrt{r^2 - x^2}}}\right] =$$

$$= \pi \cdot \left[\sqrt{2r^2 + 2r\sqrt{r^2 - x^2}} - \frac{\dfrac{r\,x^2}{\sqrt{r^2 - x^2}}}{\sqrt{2r^2 + 2r\sqrt{r^2 - x^2}}}\right] = \pi \cdot \frac{2r^2 + 2r\sqrt{r^2 - x^2} - \dfrac{r\,x^2}{\sqrt{r^2 - x^2}}}{\sqrt{2r^2 + 2r\sqrt{r^2 - x^2}}} =$$

$$= \pi \cdot \frac{\dfrac{2r^2\sqrt{r^2 - x^2} + 2r \cdot \left(r^2 - x^2\right) - r\,x^2}{\sqrt{r^2 - x^2}}}{\sqrt{2r^2 + 2r\sqrt{r^2 - x^2}}} = \pi \cdot \frac{\dfrac{2r^2\sqrt{r^2 - x^2} + 2r^3 - 2rx^2 - r\,x^2}{\sqrt{r^2 - x^2}}}{\sqrt{2r^2 + 2r\sqrt{r^2 - x^2}}} =$$

$$= \pi \cdot \frac{2r^2\sqrt{r^2 - x^2} + 2r^3 - 3rx^2}{\sqrt{r^2 - x^2} \cdot \sqrt{2r^2 + 2r\sqrt{r^2 - x^2}}} = \pi \cdot \frac{2r^2\sqrt{r^2 - x^2} + 2r^3 - 3rx^2}{\sqrt{\left(r^2 - x^2\right) \cdot \left(2r^2 + 2r\sqrt{r^2 - x^2}\right)}}$$

La derivata prima $S'(x) = 0$ per:

$$\pi \cdot \frac{2r^2\sqrt{r^2 - x^2} + 2r^3 - 3rx^2}{\sqrt{\left(r^2 - x^2\right) \cdot \left(2r^2 + 2r\sqrt{r^2 - x^2}\right)}} = 0\,; \qquad 2r^2\sqrt{r^2 - x^2} + 2r^3 - 3rx^2 = 0\,;$$

$$\left(2r^2\sqrt{r^2 - x^2}\right)^2 = \left(3rx^2 - 2r^3\right)^2; \qquad 4r^4 \cdot \left(r^2 - x^2\right) = 9r^2x^4 + 4r^6 - 12r^4x^2\,;$$

$$4r^6 - 4r^4x^2 - 9r^2x^4 - 4r^6 + 12r^4x^2 = 0\,; \qquad -9r^2x^4 + 8r^4x^2 = 0\,; \qquad x^2 \cdot \left(-9r^2x^2 + 8r^4\right) = 0\,;$$

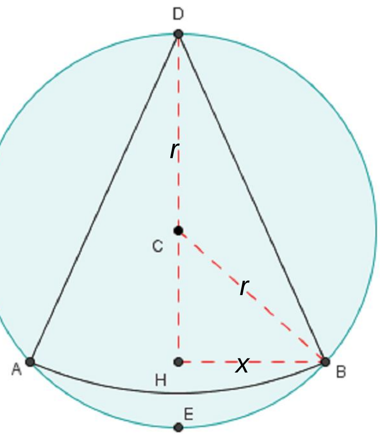
$$\begin{cases} x^2 = 0 \\ -9r^2x^2 + 8r^4 = 0 \end{cases} \qquad \begin{matrix} x^2 = 0 & x = 0 \ \ \textit{doppia} \\ x^2 = \dfrac{8r^4}{9r^2} & x = \mp\dfrac{2\sqrt{2}}{3}r \ \ \textit{la soluzione negativa non è accettabile} \end{matrix}$$

Essendo:

$S(0) = \pi \cdot 0 \cdot \sqrt{2r^2 + 2r\sqrt{r^2 - 0^2}} = 0$ $S(\pi) = \pi \cdot r \cdot \sqrt{2r^2 + 2r\sqrt{r^2 - r^2}} = \sqrt{2}\, \pi\, r^2$ $S\left(\frac{2\sqrt{2}}{3}r\right) = \pi \cdot \frac{2\sqrt{2}}{3}r \cdot \sqrt{2r^2 + 2r\sqrt{r^2 - \left(\frac{2\sqrt{2}}{3}r\right)^2}} =$ $= \pi \cdot \frac{2\sqrt{2}}{3}r \cdot \sqrt{2r^2 + 2r\sqrt{r^2 - \frac{8}{9}r^2}} =$ $= \pi \cdot \frac{2\sqrt{2}}{3}r \cdot \sqrt{2r^2 + 2r \cdot \frac{r}{3}} = \pi \cdot \frac{2\sqrt{2}}{3}r \cdot \sqrt{\frac{8}{3}r^2} =$ $= \pi \cdot \frac{8}{3\sqrt{3}}r^2 = \frac{8}{9}\sqrt{3}\, \pi\, r^2$	$\Rightarrow$	Il massimo assoluto è $M = \frac{8}{9}\sqrt{3}\, \pi\, r^2$ assunto nel punto $x = \frac{2\sqrt{2}}{3}r$

Si può concludere quindi che il cono avente la superficie laterale massima ha il raggio di base uguale a $x = \frac{2\sqrt{2}}{3}r$ **.**

Soluzione 2



Posto $\overline{BH} = x$ con $x \in [0,\ r]$

si ha: $\overline{CH} = \sqrt{r^2 - x^2}$ $\qquad \overline{DH} = r + \sqrt{r^2 - x^2}$

$$\overline{BD} = \sqrt{\overline{DH}^2 + \overline{BH}^2} = \sqrt{\left(r + \sqrt{r^2 - x^2}\right)^2 + x^2}$$

$$= \sqrt{r^2 + r^2 - x^2 + 2r\sqrt{r^2 - x^2} + x^2} = \sqrt{2r^2 + 2r\sqrt{r^2 - x^2}}$$

La funzione da rendere massima è:

$$S(x) = \frac{1}{2} \cdot 2\pi \cdot \overline{BH} \cdot \overline{BD}; \qquad \boxed{S(x) = \pi\, x \cdot \sqrt{2r^2 + 2r\sqrt{r^2 - x^2}}}$$

Agli estremi $x = 0$ e $x = r$ il cono degenera in un segmento (il diametro) avente superficie nulla.

La derivata prima è: $S'(x) = \pi \cdot \left[1 \cdot \sqrt{2r^2 + 2r\sqrt{r^2 - x^2}} + x \cdot \dfrac{2r \cdot \dfrac{-2x}{2\sqrt{r^2 - x^2}}}{2\sqrt{2r^2 + 2r\sqrt{r^2 - x^2}}}\right] =$

$$= \pi \cdot \left[\sqrt{2r^2 + 2r\sqrt{r^2 - x^2}} - \frac{\dfrac{r\,x^2}{\sqrt{r^2 - x^2}}}{\sqrt{2r^2 + 2r\sqrt{r^2 - x^2}}}\right] = \pi \cdot \frac{2r^2 + 2r\sqrt{r^2 - x^2} - \dfrac{r\,x^2}{\sqrt{r^2 - x^2}}}{\sqrt{2r^2 + 2r\sqrt{r^2 - x^2}}} =$$

$$= \pi \cdot \frac{\dfrac{2r^2\sqrt{r^2 - x^2} + 2r \cdot (r^2 - x^2) - r\,x^2}{\sqrt{r^2 - x^2}}}{\sqrt{2r^2 + 2r\sqrt{r^2 - x^2}}} = \pi \cdot \frac{\dfrac{2r^2\sqrt{r^2 - x^2} + 2r^3 - 2rx^2 - r\,x^2}{\sqrt{r^2 - x^2}}}{\sqrt{2r^2 + 2r\sqrt{r^2 - x^2}}} =$$

$$= \pi \cdot \frac{2r^2\sqrt{r^2 - x^2} + 2r^3 - 3rx^2}{\sqrt{r^2 - x^2} \cdot \sqrt{2r^2 + 2r\sqrt{r^2 - x^2}}} = \pi \cdot \frac{2r^2\sqrt{r^2 - x^2} + 2r^3 - 3rx^2}{\sqrt{(r^2 - x^2) \cdot \left(2r^2 + 2r\sqrt{r^2 - x^2}\right)}}$$

La derivata prima $S'(x) = 0$ per:

$$\pi \cdot \frac{2r^2\sqrt{r^2 - x^2} + 2r^3 - 3rx^2}{\sqrt{(r^2 - x^2) \cdot \left(2r^2 + 2r\sqrt{r^2 - x^2}\right)}} = 0; \qquad 2r^2\sqrt{r^2 - x^2} + 2r^3 - 3rx^2 = 0;$$

$$\left(2r^2\sqrt{r^2 - x^2}\right)^2 = \left(3rx^2 - 2r^3\right)^2; \qquad 4r^4 \cdot (r^2 - x^2) = 9r^2x^4 + 4r^6 - 12r^4x^2;$$

$$4r^6 - 4r^4x^2 - 9r^2x^4 - 4r^6 + 12r^4x^2 = 0; \qquad -9r^2x^4 + 8r^4x^2 = 0;$$

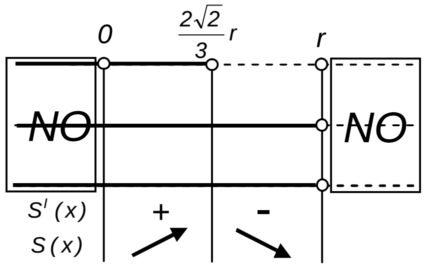
$$x^2 \cdot \left(-9r^2x^2 + 8r^4\right) = 0; \qquad \left\langle \begin{array}{l} x^2 = 0 \\ -9r^2x^2 + 8r^4 = 0 \end{array} \right. \qquad \begin{array}{l} x^2 = 0 \\ x^2 = \dfrac{8r^4}{9r^2} \end{array} \qquad \begin{array}{l} x = 0 \ \text{doppia} \\ x = \mp\dfrac{2\sqrt{2}}{3}r \end{array}$$

$$S'(x) > 0 ; \qquad \pi \cdot \frac{2r^2\sqrt{r^2-x^2}+2r^3-3rx^2}{\sqrt{\left(r^2-x^2\right)\cdot\left(2r^2+2r\sqrt{r^2-x^2}\right)}} > 0 ;$$

$$2r^2\sqrt{r^2-x^2}+2r^3-3rx^2 > 0 \quad (*) \qquad x < \frac{2\sqrt{2}}{3}r$$

$$r^2-x^2 > 0 \qquad -r < x < +r$$

$$2r^2+2r\sqrt{r^2-x^2} > 0 \quad (**) \qquad -r < x < +r$$



Si può concludere quindi che il cono avente la superficie laterale massima ha il raggio di base uguale a $x = \frac{2\sqrt{2}}{3}r$.

Calcoli

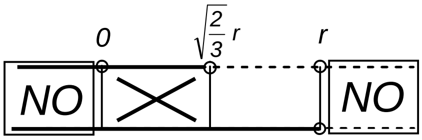
La disequazione (*)

$2r^2\sqrt{r^2-x^2}+2r^3-3rx^2 > 0$; dividendo per $2r^2 > 0$ si ha: $\sqrt{r^2-x^2}+r-\frac{3x^2}{2r} > 0$; cioè:

$$\sqrt{r^2-x^2} > \frac{3}{2r}x^2-r$$

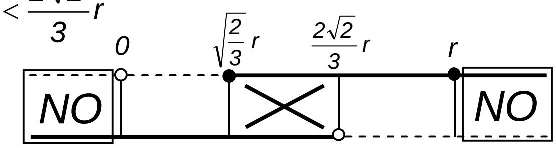
La soluzione di tale equazione è data dall'unione dei due sistemi: A $\begin{cases} \frac{3}{2r}x^2-r<0 \\ r^2-x^2 \ge 0 \end{cases}$ $\cup$ B $\begin{cases} \frac{3}{2r}x^2-r \ge 0 \\ r^2-x^2 > \left(\frac{3}{2r}x^2-r\right)^2 \end{cases}$

A $\begin{cases} \frac{3}{2r}x^2-r<0 \\ r^2-x^2 \ge 0 \end{cases}$ $\quad \begin{cases} -\sqrt{\frac{2}{3}}\,r < x < +\sqrt{\frac{2}{3}}\,r \\ -r \le x \le +r \end{cases}$



$0 < x < +\sqrt{\frac{2}{3}}\,r$

B $\begin{cases} \frac{3}{2r}x^2-r \ge 0 \\ r^2-x^2 > \left(\frac{3}{2r}x^2-r\right)^2 \end{cases}$ $\quad \begin{cases} x \le -\sqrt{\frac{2}{3}}\,r ; \quad x \ge +\sqrt{\frac{2}{3}}\,r \\ 0 < x < \frac{2\sqrt{2}}{3}r \quad (\#) \end{cases}$ $\quad \sqrt{\frac{2}{3}}\,r \le x < \frac{2\sqrt{2}}{3}r$

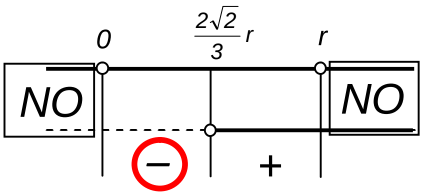


Pertanto $(**)$ $\quad 2r\sqrt{r^2-x^2}+2r^2-3x^2 > 0$ per $x < \frac{2\sqrt{2}}{3}r$.

La disequazione $(\#)$ $r^2-x^2 > \left(\frac{3}{2r}x^2-r\right)^2$; $\quad r^2-x^2 > \frac{9}{4r^2}x^4+r^2-3x^2$; $\quad \frac{9}{4r^2}x^4-2x^2<0$; $\quad x^2\cdot\left(\frac{9}{4r^2}x^2-2\right)<0$

$x^2 > 0 \qquad \forall x \ne 0$

$\frac{9}{4r^2}x^2-2>0 \qquad x < -\frac{2\sqrt{2}}{3}r ; \quad x > +\frac{2\sqrt{2}}{3}r$



$0 < x < \frac{2\sqrt{2}}{3}r$

La disequazione (**)

$2r^2+2r\sqrt{r^2-x^2} > 0$; $\quad$ essendo $r>0$ $r^2-x^2>0$; $\quad -r<x<+r$.